Surface Gravity Wave Groups Drive "Stokes Pumping" and Radiate Internal Gravity Wave Wakes

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Motivating Questions

Surface wave (SW) groups drive a deep circulation which we call the Eulerian return flow. In the presence of stable stratification, the Eulerian return flow sets isopycnals into motion, generating an internal wave (IW) wake. This sets up a system in which SWs are an energy source, and IWs are an energy sink, leading us to ask:

1. Do the radiated IWs appreciably dampen SW groups?
2. Is the energy supplied by SWs a significant source for IW generation relative to other sources in the ocean?

Introduction to Surface Wave Groups

Surface gravity waves induce a Lagrangian mean flow that is commonly known as Stokes drift. If the surf wave (SW) amplitude varies in time and space, as is the case with groups, so does the amplitude of the Stokes drift. Surface wave solutions are found by assuming small wave slopes ($\epsilon = a_{\max}k$) to linearize and solve the Boussineq equations

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = b\hat{\boldsymbol{z}}\,, \qquad (1)$$

$$b_t + \boldsymbol{u}\cdot\nabla b + wN^2 = 0\,, \qquad (2)$$

$$\nabla\cdot\boldsymbol{u} = 0\,. \qquad (3)$$

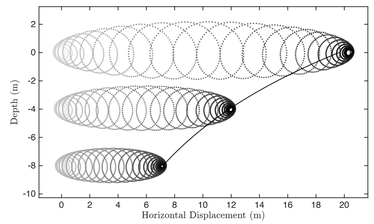


Figure 1: Numerical computation of the displacement of three example particles beneath a SW group (dots). Darker dots indicate later times, with equal spacing in time. The solid line indicates the time integrated Stokes drift which is a second order approximation to the numerical solution.

From the $O(\epsilon^1)$ solutions for velocity $\boldsymbol{u}_1 = (u_1, v_1, w_1)$ and displacement $\boldsymbol{\xi}_1 = (\xi_1, \eta_1, \zeta_1)$, we can compute the $O(\epsilon^2)$ Stokes drift

$$u^{\mathrm{S}} \overset{\text{def}}{=} \overline{\boldsymbol{\xi}_1\cdot\nabla u_1} = c\,k^2|a(x,y,t)|^2 e^{2kz}\,. \qquad (4)$$

The mean wave momentum, M, per unit area is defined as

$$M(x,y,t) \overset{\text{def}}{=} \overline{u_1\zeta_1}\big|_0 = \tfrac{1}{2}ck\,|a|^2 = \int_{-\infty}^{0} u^{\mathrm{S}}\,\mathrm{d}z\,. \qquad (5)$$

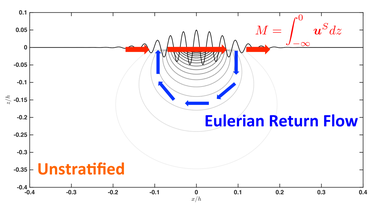


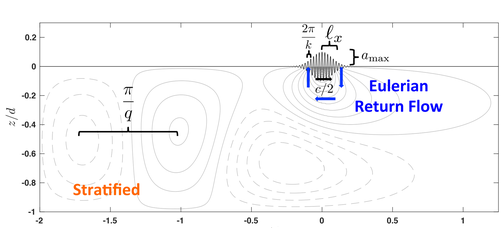
Figure 2: Eulerian return flow in an unstratified ocean.

Convergence and divergence in M drives water downward ahead of the group, and lifts it in the rear. We call this forcing "Stokes pumping," and it sets the surface boundary condition on vertical velocity.

$$\text{at } z = 0: \qquad \bar{w}_2 \approx M_x\,. \qquad (6)$$

Scales of Interest

- SW amplitude $a_{\max} \sim 1\text{m}$
- SW wavenumber $k \sim \frac{2\pi}{100\text{m}}$
- SW period $T \sim 8\text{s}$
- SW phase speed $c \sim 12.5\text{ms}^{-1}$
- SW group length $\ell_x \sim 250\text{m}$
- SW group width $\ell_y \sim 3\ell_x$
- Depth $d \sim 2000\text{m}$
- Buoyancy frequency $N \sim \frac{2\pi}{2000\text{s}}$
- IW z-wavenumber $m_n = \frac{n\pi}{d}$
- IW y-wavenumber $s \sim q$
- IW x-wavenumber $q \sim q_{\max} = \frac{2N}{c} = \frac{2\pi}{12.5\text{km}}$



Resonance and Internal Wave Radiation

Combining (1)-(3) and performing a phase average ($\overline{\cdot}$) over the SW timescale we find an $O(\epsilon^2)$ equation for the vertical velocity of IWs

$$\left[\partial_t^2\nabla^2 + N^2\left(\partial_x^2 + \partial_y^2\right)\right]\bar{w}_2 = 0\,. \qquad (7)$$

To solve (7) we project onto n vertical modes and Fourier transform ($\hat{\cdot}$) in the horizontal to find

$$\hat{w}_n = \frac{-\sqrt{2}m_n}{d}\,\frac{\mathrm{i}q\hat{M}(q,s)}{\left(|\mathbf{q}|^2 + m_n^2\right) - \left(\frac{q_{\max}}{q}\right)^2|\mathbf{q}|^2}\,, \qquad (8)$$

where $\mathbf{q} = (q, s)$. The inverse Fourier transform of (8) gives the vertical velocity of the IW wake plotted in figure 3.

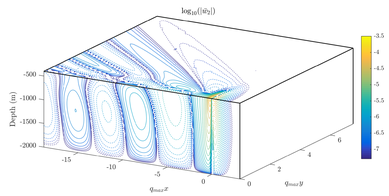


Figure 3: Common logarithm of the vertical velocity $\bar{w}_2$. Dashed contours indicate negative values.

Resonance occurs, and IWs are radiated, when the denominator of (8) vanishes giving us the resonance condition

$$\underbrace{\frac{c}{2}}_{\text{SW group speed}} = \underbrace{\frac{N}{q}\sqrt{\frac{q^2+s^2}{q^2+s^2+m_n^2}}}_{\text{IW phase speed in } x\text{-direction}}\,. \qquad (9)$$

From the resonance condition we can deduce the widest possible wake angle

$$\max_{\forall(q,s)}(\sin\theta_n) = \frac{2N}{cm_n} = \frac{2Nd}{n\pi c}\,, \qquad (10)$$

as depicted in figure 4

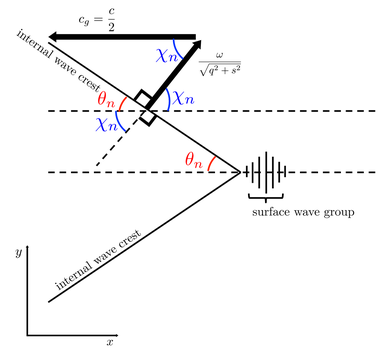


Figure 4: Schematic of the IW wake. In the frame moving with the SW group, the surrounding water rushes backward at the SW group speed $c/2$.

Energy Flux from SW to IW

To compute the energy transfer from SWs to IWs we compute the vertical energy flux through the surface

$$J = \sum_{n=1}^{\infty}\underbrace{\iint_{-\infty}^{\infty} M_x\,\varpi_n\big|_0\,\mathrm{d}x\mathrm{d}y}_{\overset{\text{def}}{=}J_n}\,. \qquad (11)$$

In (11) we have replaced $\bar{w}_2$ with the boundary condition (6), and $\varpi_n|_0$ the pressure for the nth mode at the surface can be computed by eliminating the buoyancy from (1) and (2) and then using (8).

Energy Flux Scale Dependence

Exact

$$J_n = -\frac{c}{2}\frac{1}{\sqrt{2}\pi d}\int_0^{q_{\max}} q^2\left|\hat{M}(q, s_n(q))\right|^2\sqrt{\frac{q_{\max}^2 - q^2}{m_n^2 - q_{\max}^2 + q^2}}\,\mathrm{d}q \qquad (12)$$

If SWs are fast, stratification is weak, and groups are wider than long.

$$J \approx J_1 \approx -\frac{32\sqrt{2}\pi^5}{g^3}\frac{(Na_{\max})^4}{T^5}(\ell_x\ell_y)^2\int_0^1 q^2\exp\left[-\frac{q_*^2}{1-q_*^2}\frac{(m_n\ell_y)^2}{2}\right]\sqrt{1-q_*^2}\,\mathrm{d}q_* \qquad (13)$$

If SWs are fast, stratification is weak, and groups are wider than the depth

$$J \approx -\frac{32\pi^{5/2}d^3(Na_{\max})^4\ell_x^2}{g^3\,T^5\ell_y}\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^3}}_{\approx 1.2} \qquad (14)$$

Wave Group Examples

To see this effect we examine a SW group with Gaussian amplitude modulation

$$a = a_{\max}\exp\left(-\frac{\left(x - \frac{c}{2}t\right)^2}{2\ell_x^2} - \frac{y^2}{2\ell_y^2}\right)\,, \qquad (15)$$

Figure 5 shows how the energy flux is distributed across different IW wavenumbers. The majority of the energy goes into near-N internal waves, and into the first vertical mode $n = 1$.

Table 1 shows that for typical and even extreme SW forcing, the radiation of IWs does not appreciably dampen the SWs. In general this mechanism is also a very weak source of energy for IWs. Pinkel (1975) observed a near-N spectral peak of IW energy. Comparing the energy flux from the extreme forcing case to the energy contained in this spectral peak suggests that extreme SW forcing over a period of one day would generate the observed amount of energy in the near-N spectral peak.

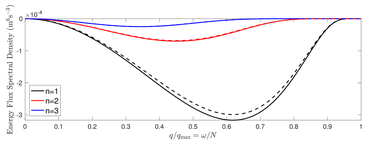


Figure 5: Energy flux spectral density for a Gaussian wave group (15). The first three modes are plotted. The solid lines are the exact solution (12), and the dashed lines are the approximate solution (13).

Table 1: Parameters for typical and extreme forcing, and results for the energy flux computed using (12).

Parameter	Typical Forcing	Extreme Forcing
wavenumber k	$2\pi/(100\,\text{m})$	$2\pi/(625\,\text{m})$
wave amplitude $a_{\max}$	1 m	4 m
group length $\ell_x = \pi n_{SW}/k$	250 m	1.56 km
group width ℓ_y	$3\ell_x = 750$ m	$3\ell_x = 4.69$ km
buoyancy frequency N	$2\pi/(2000\,\text{s})$	$2\pi/(1333\,\text{s})$
$q_{\max} = 2N/c$	$2\pi/(12.5\,\text{km})$	$2\pi/(20.8\,\text{km})$
Energy flux J	0.2W	100W
Half-life of SW	many years	many years
Time to force near-N spectral peak	500 days	1 day

Conclusions

- Stokes pumping heaves isopycnals below, causing the radiation of IWs.
- Radiated IWs have frequencies near N and a maximum wavenumber determined by the speed of the SW group and the stratification.
- For IWs to radiate, the IW phase speed in x-direction must match the SW group speed.
- The energy flux from SWs to IWs depends strongly on the SW amplitude and period, and the stratification [see (12)-(14)]
- Radiated IWs do not appreciably dampen SWs and SWs are a weak source of IW energy.
- Extreme SW forcing over a period of one day is strong enough to account for the observed spectral peak near the buoyancy frequency.

References

- Pinkel, R. 1975 Upper ocean internal wave observations from flip. Journal of Geophysical Research 80 (27), 3892 – 3910.
- Corresponding manuscript: S. Haney, W.R. Young, Radiation of internal gravity waves by groups of surface gravity waves: 2017. Journal of Fluid Mechanics, in reveiw.

This manuscript and other works can be found at https://seanrhaney.github.io/